ON BERGMAN'S PROPERTY FOR THE AUTOMORPHISM GROUPS OF RELATIVELY FREE GROUPS

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1. Introduction

Let Ω be an infinite set and $\mathrm{Sym}(\Omega)$ the full symmetric group on Ω. In his recent preprint [1] Bergman proved the following delightful result. Consider a system $(Y_i : i \in I)$ where $|I| \leqslant |\Omega|$ of *subsets* of $\mathrm{Sym}(\Omega)$ whose union is $\mathrm{Sym}(\Omega)$. Then there is a member $Y = Y_{i_0}$ of the system such that letting Z denote the conjugate set $\pi Y \pi^{-1}$, where π is a suitable involution of $\mathrm{Sym}(\Omega)$ we have that

$$\mathrm{Sym}(\Omega) = (YZ)^4 Y \cup (ZY)^4 Z. \tag{1.1}$$

This easily implies that the confinality of $\mathrm{Sym}(\Omega)$ is greater than $|\Omega|$, the result first proven by Macpherson and Neumann in [15] (the *confinality* of a given group G being the least cardinality of a chain of proper subgroup whose union is G.) Another consequence of the result of Bergman's is really surprising! One of the examples of systems (Y_i) is given by the system of powers $(X^m : m \in \mathbf{N})$ of a generating set X of $\mathrm{Sym}(\Omega)$ which is closed under taking inverses. By a *power* X^m of X we mean the set of all products of the form $x_1 \dots x_m$, where $x_i \in X$. Then it follows from (1.1) that $\mathrm{Sym}(\Omega)$ equals X^k for some natural number k. This naturally leads to the following definition.

Let G be a group. We say that G has *Bergman's property*, if for every generating set S of G with $S^{-1} = S$ we have that $G = S^k$ for some natural number k [10, 9]; it might be also said that G is a *group of universally finite width*. Recall that the *width* of a group G with respect to a generating set S with $S = S^{-1}$ is the least natural number k such that $G = S^k$, or ∞, otherwise.

The first example of an infinite group of universally finite width had been found apparently by Shelah in [20]. The paper [20] contains an example of an uncountable group G such that the width of G relative to any generating set is at most 240. The striking results of Bergman's and especially his inspiring remark in the first draft of [1] that the further examples of the kind might be found among 'the automorphism groups of other structures that can be put together out of many isomorphic copies of themselves' gave rise to the search of other examples of groups of universally finite width. Bergman himself suggested to analyze the situation with the automorphism group of $\mathbf{R}$ as a Borel space, the automorphism group of homogeneous Boolean spaces, the infinite-dimensional general linear groups, the automorphism groups of infinitely generated free groups and some other automorphism groups.

A short while later Droste–Holland [10] have proved that the automorphism group of any doubly homogeneous chain has Bergman's property, Droste–Göbel [9] have settled Bergman's question about the automorphism group of $\mathbf{R}$ as a Borel space establishing that this group is a group of universally finite width and the infinite-dimensional general linear groups over divisions rings also turned out to have Bergman's property [21].

Basing on (1.1), Droste and Göbel came to the notion of strong confinality of a group. Let G be a group. Consider a *chain* of subsets $(Y_i : i \in I)$ of G such that

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$$\text{for all } i \in I \text{ the set } Y_i \text{ is a proper subset of } G \text{ such that } Y_i^{-1} = Y_i \text{ and there exists } j = j(i) \in I \text{ such that } Y_i Y_i \subseteq Y_j. \tag{1.1}$$

Then the *strong confinality* of G is the least cardinality of a chain $(Y_i : i \in I)$ with (1.1) such that $G = \bigcup_{i \in I} Y_i$. Bergman's results immediately imply that the strong confinality of the full symmetric group $\mathrm{Sym}(\Omega)$, where Ω is an infinite set is greater than $|\Omega|$.

The notions of confinality, strong confinality and Bergman's property are embraced by the following observation from [10]: a group G has uncountable strong confinality if and only if it has uncountable confinality and Bergman's property. The group

$$\mathrm{BSym}(\mathbf{Q}) = \{g \in \mathrm{Sym}(\mathbf{Q}) : \exists k \in \mathbf{N} \text{ s.t. } \forall x \in \mathbf{Q}\, |gx - x| \leqslant k\}$$

of bounded permutations of $\mathbf{Q}$ provides a nice illustration to this observation: it has uncountable confinality, but not Bergman's property [9, Theorem 3.6].

Bergman's property can be also applied for analysis of the structural properties of groups and for answering questions whether a given group belongs or does not belong to a certain class of groups. For example, one of the questions considered in [16] is whether a Polish group with a comeagre conjugacy class (the infinite symmetric groups are examples of such groups, by [22]) can be written non-trivially as a free product with amalgamation. Bergman's result provides immediately a negative answer for $\mathrm{Sym}(\Omega)$ on an infinite set Ω (which, before [16], was known only for the case when Ω is countable, [19]): $\mathrm{Sym}(\Omega)$ is a group of universally finite width, whereas any nontrivial free product $G_1 *_A G_2$ with amalgamation has infinite width relative to the generating set $G_1 \cup G_2$

In the present paper we consider the problem of universality of finite width for the automorphism groups of relatively free groups. We prove that the automorphism group $\mathrm{Aut}(N)$ of any infinitely generated free nilponent group N is a group of universally finite width (and so does any homomorphic image of $\mathrm{Aut}(N)$.) We give also a partial answer to Bergman's question about the automorphism group $\mathrm{Aut}(F)$ of infinitely generated free group F: in the case when F is countable, the group $\mathrm{Aut}(F)$ is indeed a group of universally finite width.

Suppose that Γ is a permutation group that acts on a set Ω. We shall use standard permutation notation, extending it, as in [1], to arbitrary subsets of Γ. Thus, if Y is a subset of Γ and U is a subset of Ω, $Y_{(U)}$ is the set of all elements of Y that fix U pointwise and $Y_{\{U\}}$ is the set of all elements of Y that fix U setwise. Any notation like $Y_{*_1,*_2}$ means the set $Y_{*_1} \cap Y_{*_2}$.

The paper is organized as follows. In Section 1 we study the automorphisms groups of relatively free algebras. Suppose that $\mathbf{V}$ is a variety algebras and F is a free algebra of $\mathbf{V}$ of infinite rank. Write Γ for $\mathrm{Aut}(F)$. Our goal is to prove that for any Droste-Göbel chain $(H_i : i \in I)$ with $\Gamma = \bigcup_i H_i$ of cardinality at most rank F there is a member H_{i_0} of (H_i) such that H_{i_0} contains a stabilizer $\Gamma_{(U),\{W\}}$, where $F = U * W$ is the free product of subalgebras U and W that are both isomorphic to F (Lemma 2.1). Lemma 2.1 is then repeatedly used in the proofs of the results on universality of finite width for the automorphism group of relatively free groups we have mentioned above.

The reader may remember that a matter of generation of a group under consideration by a pair of stabilizers of the form $\Gamma_{(U),\{W\}}$ is one of the central themes in many papers on confinalities (the small index property) of certain automorphisms groups Γ. With this idea in mind, we introduce the following definition. We call a variety $\mathbf{V}$ of algebras a *BMN-variety* if for every free algebra F of infinite rank of $\mathbf{V}$ we have that $\Gamma = \mathrm{Aut}(F)$ is generated by any pair of stabilizers

$$\Gamma_{(U_1),\{U_2 * W\}} \text{ and } \Gamma_{(U_2),\{U_1 * W\}}, \tag{1.2}$$

where $F = U_1 * U_2 * W$ and free factors U_1, U_2, W are all isomorphic to F. The results from [8] and [14] can interpreted in effect that the variety of all sets with no structure and any variety of vector spaces over a fixed field are BMN-varieties. With the use of Lemma 2.1 it is rather easy to show that the automorphism group $\mathrm{Aut}(F)$ of a free algebra F of infinite rank in a BMN-variety has Bergman's property if and only if the width of $\mathrm{Aut}(F)$ relative to any generating set of the

form (1.2) is finite. Also, rather simple arguments show that $\mathrm{Aut}(F)$ is generated by involutions, is perfect and has confinality greater than rank F.

In Section 2 we show that variety $\mathfrak{A}$ of all abelian groups and any variety $\mathfrak{N}_c$ of nilpotent groups of class $\leqslant c$ are BMN-varieties and that the automorphism groups of relatively free groups in these varieties have Bergman's property. The abelian case is based upon an impressive result by Swan which says, roughly speaking, that the automorphism group $\mathrm{Aut}(A)$ of a free abelian group A of infinite rank has finite width relative to the set of all automorphisms of A that have 'unimodular matrices' with regard to a fixed basis of A. The nilpotent case is then proved by induction on c.

Methods of Section 3 are different: here we apply a powerful fact from a paper [2] by Bryant–Evans stating that a free group F of countably infinite rank possesses automorphisms that are generic with respect to the family of all finitely generated free factors of F. Then a modification of the famous binary tree argument from a paper [11] by Hodges–Hodkinson–Lascar–Shelah shows that $\mathrm{Aut}(F)$ has Bergman's property. The scope of application of methods from [2, 11] is restricted by Polish groups, and so the general case of Bergman's question about the automorphism groups of infinitely generated free groups remains unsettled.

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2. Defining a class of well-behaved varieties of algebras

Let $\mathfrak{L}$ be a language that consists only of functional symbols. Then an $\mathfrak{L}$-*algebra* is a structure in the language $\mathfrak{L}$. An $\mathfrak{L}$-*variety* is a class of all $\mathfrak{L}$-algebras satisfying a certain set of identities over $\mathfrak{L}$. *Free algebras* of an $\mathfrak{L}$-variety, their *bases* and *ranks* are defined in the usual way of universal algebra [7].

Let $\mathbf{V}$ be a variety of algebras and F a free algebra of $\mathbf{V}$ of infinite rank. Write Γ for the automorphism group of F. Similarly to [14] we call a free factor U of F *moietous*, if U is isomorphic to F and for some decomposition

$$F = U * W,$$

where $*$ denotes the free product of algebras, the free factor W is also isomorphic to F. In particular,

$$\text{rank } F = \text{rank } U = \text{rank } W = \text{corank } U = \text{corank } W.$$

Assume that

$$F = \prod_{i\in I}^{*} U_i$$

is a decomposition of F into free factors and σ_i is an automorphism of a free factor U_i $(i \in I)$; then

$$\prod_{i\in I}^{*} \sigma_i$$

denotes the only automorphism σ of F which extends all the automorphisms σ_i.

The following result, the key result of the section, generalizes the similar result from [21].

Lemma 2.1. *Let $\mathbf{V}$ be a variety of algebras, F a free algebra of $\mathbf{V}$ of infinite rank and Γ denote the automorphism group $\mathrm{Aut}(F)$. Assume that $\Gamma = \bigcup_{i\in I} H_i$, where $|I| \leqslant \text{rank } F$ and $(H_i : i \in I)$ is a chain of proper subsets of Γ all closed under inverses and such that for every $i \in I$ there is $j \in I$ with $H_iH_i \subseteq H_j$. Then an appropriate member H_k of the chain (H_i) contains a subgroup $\Gamma_{(U),\{W\}}$, where U, W are moietous free factors of F with $F = U * W$.*

Proof. First, we apply a certain 'diagonal' argument to see that there exists

a member H_{i_0} of $(H_i : i \in I)$ such that for some moietous free factors U, W of F with $F = U * W$ the set $(H_{i_0})_{\{U\},\{W\}}$ induces the full automorphism group $\mathrm{Aut}(U)$ on U. (2.1)

Let

$$F = \prod_{i \in I}^{*} V_i$$

be a decomposition of F into moietous free factors. Write

$$V_k^{\times} = \prod_{i \neq k}^{*} V_i$$

for all $k \in I$.

If for some pair (H_k, V_j)

$$(H_k)_{\{V_j\},\{V_j^{\times}\}} \text{ induces } \mathrm{Aut}(V_j) \text{ on } V_j,$$

we are done. Suppose otherwise. Then, in particular, for all i in I

$$(H_i)_{\{V_i\},\{V_i^{\times}\}} \text{ does not induce } \mathrm{Aut}(V_i) \text{ on } V_i.$$

Hence for each $i \in I$ we can find $\sigma_i \in \mathrm{Aut}(V_i)$ such that

$$\sigma_i \text{ does not equal to the restriction on } V_i \text{ of any element from } (H_i)_{\{V_i\},\{V_i^{\times}\}}.$$

Set

$$\sigma = \prod_{i \in I}^{*} \sigma_i.$$

Since $\Gamma = \mathrm{Aut}(F) = \bigcup_{i \in I} H_i$, we have $\sigma \in H_m$ for some $m \in I$. It is clear, however, that

$$\sigma \in (H_m)_{\{V_m\},\{V_m^{\times}\}}.$$

But then the restriction of σ on V_m is σ_m, which is impossible.

So let H_{i_0}, U, W satisfy the condition (2.1). We fix some free decomposition of U into moietous free factors

$$U = \prod_{k \in \mathbf{N}}^{*} U_k$$

and for each $k \in \mathbf{N}$ choose a basis $\mathcal{B}_k$ of U_k. Let $U_0^{\times}$ denote the free factor

$$\prod_{k \neq 0}^{*} U_k$$

Consider also a family of bijections $f_{rs} : \mathcal{B}_r \to \mathcal{B}_s$, where r, s with $r < s$ run over $\mathbf{N}$.

We prove that a member of $(H_i : i \in I)$ contains the group $\Gamma_{\{U_0\},(U_0^{\times} * W)}$. Suppose α is any automorphism of U_0; write $\alpha_{(k)}$ for the automorphism $f_{0k} \alpha f_{0k}^{-1}$ of a free factor U_k. Consider the family of all automorphisms in H_{i_0} that have the form

$$\gamma = \alpha * \prod_{n \geqslant 1}^{*} \alpha_{(3n)}^{-1} * \prod_{n \geqslant 0}^{*} \alpha_{(3n+1)} * \prod_{n \geqslant 0}^{*} \mathrm{id}_{U_{3n+2}} * \beta \tag{2.2}$$

where α is an (arbitrary) automorphism of U_0, β is an automorphism of W and an automorphism in (2.2) is constructed over the free decomposition

$$F = U_0 * \prod_{n \geqslant 1}^{*} U_{3n} * \prod_{n \geqslant 0}^{*} U_{3n+1} * \prod_{n \geqslant 0}^{*} U_{3n+2} * W.$$

In a more simple though less formal way any automorphism of the form (2.2) can be written as

$$\gamma = \alpha * (\alpha^{-1} * \alpha^{-1} * \ldots) * (\alpha * \alpha * \ldots) * (\mathrm{id} * \mathrm{id} * \ldots) * \beta,$$

Then we have

$$\gamma^{-1} = \alpha^{-1} * (\alpha * \alpha * \ldots) * (\alpha^{-1} * \alpha^{-1} * \ldots) * (\mathrm{id} * \mathrm{id} * \ldots) * \beta^{-1},$$

There exists an automorphism π of F such that

(a) π acts as a permutation on $\bigcup_{r\in\mathbf{N}}\mathcal{B}_r$ and agrees on each $\mathcal{B}_r$ either with some f_{rs} or with some f_{sr}^{-1};

(b) π fixes W pointwise;

(c) and, finally, such that

$$\pi\gamma^{-1}\pi^{-1} = \mathrm{id} * (\alpha*\alpha*\ldots)*(\alpha^{-1}*\alpha^{-1}*\ldots)*(\mathrm{id}*\mathrm{id}*\ldots)*\beta^{-1},$$

This implies that

$$\gamma\pi\gamma^{-1}\pi^{-1} = \alpha*(\mathrm{id}*\mathrm{id}*\ldots)*(\mathrm{id}*\mathrm{id}*\ldots)*(\mathrm{id}*\mathrm{id}\ldots)*\mathrm{id},$$

which means that $\gamma\pi\gamma^{-1}\pi^{-1}$ is an element of $\Gamma_{\{U_0\},(U_0^{\times}*W)}$. Assume now that π,π^{-1} are both elements of some H_{j_0} in the chain $(H_i : i\in I)$. Then we have

$$\Gamma_{\{U_0\},(U_0^{\times}*W)} \subseteq H_{i_0}\pi H_{i_0}\pi^{-1} \subseteq (H_{i_0}H_{j_0})(H_{i_0}H_{j_0}) \subseteq H_j$$

for a suitable $j\in I$, and the result follows. □

Remark 2.2. In the papers [5, 6] the method we have applied for the construction of $\gamma\pi\gamma^{-1}\pi^{-1}$ is referred to as 'tricks of Whitehead and Eilenberg'.

As it has been said above pairs of stabilizers of the form $\Gamma_{(U),\{V\}}$ may generate the full automorphism group Γ. The known examples include, for instance, the infinite symmetric groups [8] and by the general linear groups of infinite-dimensional vector spaces [14]. Thus, Macpherson proves in [14] that given three moietous subspaces U_1, U_2, W of an infinite-dimensional vector space V with

$$V = U_1\oplus U_2\oplus W,$$

we have that the group $\Gamma = \mathrm{GL}(V)$ is generated by $\Gamma_{(U_1),\{U_2+W_2\}}$ and $\Gamma_{(U_2),\{U_1+W\}}$. Similarly, if

$$\Omega = U_1\cup U_2\cup W$$

is a partition of an infinite set Ω into moieties (a subset I of Ω is called a *moiety*, if $|I| = |\Omega\setminus I|$), then the group $\Gamma = \mathrm{Sym}(\Omega)$ is generated by $\Gamma_{(U_1),\{U_2\cup W\}}$ and $\Gamma_{(U_2),\{U_1\cup W\}}$ [8]. Moreover, the (very short) proof of the Lemma at page 580 in [8] shows that the following fact is true.

Lemma 2.3. *The width of the group* $\Gamma = \mathrm{Sym}(\Omega)$ *relative to the set*

$$\Gamma_{(U_2),\{U_1\cup W\}}\cup\Gamma_{(U_1),\{U_2\cup W\}}$$

is at most 3.

Very naturally, these remarkable results give rise to the following definition.

Definition. Let $\mathbf{V}$ be a variety of algebras. We say that $\mathbf{V}$ is a *BMN-variety* (Bergman-Macpherson-Neumann variety), if for every free algebra F of $\mathbf{V}$ of infinite rank given any decomposition

$$F = U_1 * U_2 * W$$

of F into moietous free factors, the groups $\Gamma_{(U_1),\{U_2*W\}}$ and $\Gamma_{(U_2),\{U_1*W\}}$ generate the full automorphism group Γ of F.

So the variety of all sets with no structure and any variety of vector spaces over a fixed division ring are examples of BMN-varieties. In the next section we shall demonstrate that the variety $\mathfrak{A}$ and any variety $\mathfrak{N}_c$ of nilpotent groups of class $\leqslant c$ are also examples of BMN-varieties. We remark also that there exists a wide class of BMN-varieties of modules.

It turns out that with the use of Lemma 2.1 we could say a good deal about the automorphism groups of free algebras of BMN-varieties.

Theorem 2.4. *Let* $\mathbf{V}$ *be a BMN-variety and* F *a free algebra of* $\mathbf{V}$ *of infinite rank. Then the full automorphism group* Γ *of* F *is a group of universally finite width if and only if the width of* Γ *relative to some (any) set of the form*

$$\Gamma_{(U_1),\{U_2*W\}} \cup \Gamma_{(U_2),\{U_1*W\}}$$

where $F = U_1 * U_2 * W$ *and* U_1, U_2, W *are moietous free factors is finite.*

Proof. The necessity part is obvious. Suppose that X is a generating set of $\Gamma = \mathrm{Aut}(F)$ which is closed under inverses; by adding the identity of Γ to X we can make it sure that the system of powers $(X^k : k \in \mathbf{N})$ is a chain. The chain $(X^k : k \in \mathbf{N})$ satisfies the conditions of Lemma 2.1 and then there are moietous free factors U, W of F with $F = U * W$ and a power X^m such that

$$X^m \supseteq \Delta = \Gamma_{(U),\{W\}}.$$

Now let

$$W = W_0 * W_1$$

be a decomposition of W into moietous free factors. Consider an automorphism ρ of F that interchanges U and W_0 and fixes W_1 pointwise. Therefore by the condition the width of Γ relative to the set

$$\Delta_1 = \Delta \cup \rho\Delta\rho^{-1}$$

is finite: for some $k \in \mathbf{N}$

$$\Gamma \subseteq \Delta_1^k.$$

Meanwhile, if $\rho \in X^l$, then

$$\Gamma \subseteq \Delta_1^k \subseteq X^{k(m+2l)},$$

and the width of Γ relative to X is finite, as desired. □

Let G be a group. Recall that the *confinality* of G is the least cardinal λ such that G can be expressed as the union of a chain of λ proper subgroups.

Theorem 2.5. *Let* $\mathbf{V}$ *be a BMN-variety,* F *a free algebra of* $\mathbf{V}$ *of infinite rank and* Γ *the automorphism group of* F*. Then*

(i) Γ *is generated by involutions;*

(ii) Γ *is perfect, that is, it coincides with the commutator subgroup:* $\Gamma = [\Gamma, \Gamma]$;

(iii) *the confinality of* $\Gamma = \mathrm{Aut}(F)$ *is greater than* rank F.

Remark 2.6. It is interesting to compare part (ii) of the lemma with the following corollary of Theorem C from a paper [4]: if a relatively free algebra G of infinite rank has the small index property, then $\mathrm{Aut}(G)$ is perfect.

Proof of Theorem 2.5. (i) Let U_1, U_2, W be three moietous free factors of F such that

$$F = U_1 * W * U_2.$$

Assume I is an index set of cardinality rank F and let

$$(a_i : i \in I), \quad (a_i^* : i \in I)$$

be bases of U_1 and $(b_i : i \in I)$ a basis of U_2. Consider two (conjugate) involutions π and π_1 which both fix W pointwise and act on the bases (a_i) and (a_i^*) as follows:

$$\pi a_i = b_i, \qquad \forall i \in I$$
$$\pi_1 a_i^* = b_i.$$

We have therefore that

$$\pi_1 \pi a_i = a_i^* \quad \forall i \in I.$$

Let now α denote the automorphism of U_1 that takes the basis (a_i) onto the basis (a_i^*). Suppose that for all $i \in I$

$$\alpha^{-1}a_i^* = t_i(\overline{a}_i^*),$$

where t_i is a reduced term in the language of $\mathbf{V}$ and $\overline{a}_i^*$ denotes a finite subset of elements of the basis $(a_i^* : i \in I)$.

We then have

$$\pi_1\pi b_i = \pi_1 a_i = \pi_1(\alpha^{-1}a_i^*) = \pi_1(t_i(\overline{a}_j^*)) = t_i(\overline{b}_j)$$

for all $i \in I$. We see that the action of $\pi_1\pi$ on $U_2 = \langle b_i : i \in I\rangle$ is isomorphic to the action of α^{-1} on $U_1 = \langle a_i^* : i \in I\rangle$, or, informally, one can write that

$$\pi_1\pi = \alpha * \mathrm{id} * \alpha^{-1}.$$

Extending the principle of the construction of $\pi_1\pi$, one can represent as a product of two conjugates of π any automorphism of F of the form

$$\prod_{n\in\mathbf{N}}{}^{*}\alpha * \mathrm{id} * \prod_{n\in\mathbf{N}}{}^{*}\alpha^{-1}, \tag{2.3}$$

where the latter product corresponds to a decomposition of F into a *moietous* free factors and α is the isomorphism type of an automorphism of one of these factors. Applying tricks of Whitehead and Eilenberg as in the proof of Lemma 2.1 we obtain that any automorphism of F of the form

$$\alpha * \mathrm{id},$$

corresponding to a decomposition of F into moietours free factors can be written as a product of two automorphisms of the form (2.3), or, in other words, this automorphism is a product of at most four conjugates of π, a product of at most four involutions.

Since $\mathbf{V}$ is a BMN-variety, the subgroups $\Gamma_{(U_1),\{U_2*W\}}$ and $\Gamma_{(U_2),\{U_1*W\}}$ generate Γ. On the other hand, each element in these subgroups is a product of at most four involutions.

(ii) The proof of (i) shows that Γ is generated by all products $\rho_1\rho_2$, where ρ_k is a conjugate of π. Clearly, any such product is a product of two commutators:

$$\rho_1\rho_2 = \sigma_1\pi\sigma_1^{-1}\sigma_2\pi\sigma_2^{-1} = \sigma_1\pi\sigma_1^{-1}\pi\cdot\pi\sigma_2\pi\sigma_2^{-1} = [\sigma_1,\pi][\pi,\sigma_2]$$

(it is also easy to see that π itself is a commutator.)

(iii) Suppose that $(H_i : i \in I)$ is a chain of proper subgroups of Γ, where $|I| \leqslant \operatorname{rank} F$ and the union of (H_i) is Γ. Then by Lemma 2.1 some member H_k of the chain contains a stabilizer $\Delta = \Gamma_{(U),\{W\}}$, where U, W are moietous free factors whose free product is F. By the condition Δ along with its appropriate conjugate $\rho\Delta\rho^{-1}$ generates Γ. Both sets Δ and $\rho\Delta\rho^{-1}$ can be found in a member, say H_m of the chain which contains H_k and ρ. Hence $H_m = \Gamma$, a contradiction. □

3. Free nilpotent groups

Proposition 3.1. (i) *The variety $\mathfrak{A}$ of all abelian groups is a BMN-variety;*

(ii) *if A is an infinitely generated free abelian group, then the automorphism group* $\mathrm{Aut}(A)$ *of A is a group of universally finite width.*

Proof. Let A be an infinitely generated free abelian group. Swan, using a very elegant transfinite induction argument [6, 5], found a set of generators of $\mathrm{Aut}(A)$ relative to which the group $\mathrm{Aut}(A)$ had finite width. More precisely, if $\mathcal{X}$ is a basis of A an automorphism θ of A is called *$\mathcal{X}$-block-unitriangular* if there is a moiety $\mathcal{Y}$ of $\mathcal{X}$ such that θ fixes $\mathcal{Y}$ elementwise and for all $z \in \mathcal{X}\setminus\mathcal{Y}$

$$\theta z \equiv z \pmod{\langle\mathcal{Y}\rangle}.$$

(in particular, the 'matrix' of θ with regard to the basis $\mathcal{Y}\cup(\mathcal{X}\setminus\mathcal{Y})$ is a block-unitriangular.)

Theorem 3.2 (Swan)**.** *The width of* Aut(A) *relative to the set of all* $\mathcal{X}$*-block-unitriangular automorphisms is at most* 22.

The proof can be found either in [6, Section 2], or in [5, Section 2].

Now let U_1, U_2, W be moietous direct summands of A with

$$A = U_1 \oplus U_2 \oplus W.$$

We show that the group Aut(A) has finite width relative to the union of the stabilizers

$$X = \Gamma_{(U_1),\{U_2+W\}} \cup \Gamma_{(U_2),\{U_1+W\}}.$$

Suppose that $\mathcal{B}_1, \mathcal{B}_2$ and $\mathcal{B}_W$ are bases of U_1, U_2 and W respectively. Write $\mathcal{B}$ for $\mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_W$. By Swan's result and Theorem 2.4 it suffices to show that the lengths of all $\mathcal{B}$-block-unitriangular automorphisms relative to X are bounded from above. (The *length* of an element g of a group G with respect to a generating set S of G is the smallest natural k such that g can be expressed a product of k elements of $S \cup S^{-1}$.)

First, note that by Lemma 2.3 any automorphism of A that acts on $\mathcal{B}$ as a permutation has the length at most 3 with respect to X. Take a $\mathcal{B}$-block-unitriangular automorphism θ. Suppose that $(x_i : i \in I)$ is a basis of U_2. A suitable conjugate $\theta' = \pi^{-1}\theta\pi$ by $\pi \in$ Aut(A) which acts on $\mathcal{B}$ as a permutation preserves all elements of $\mathcal{B}_1 \cup \mathcal{B}_W$, and for all $i \in I$ we have that

$$\theta' x_i = x_i + a_i + b_i$$

where $a_i \in W$ and $b_i \in U_1$. The proof can be now completed as in the proof of Proposition 2.2 from [14]. Consider $\tau_1 \in \Gamma_{(U_1),\{U_2+W\}}$ which fixes W pointwise and such that

$$\tau_1 x_i = x_i - a_i$$

for all $i \in I$. Clearly, for every $i \in I$

$$\tau_1\theta' x_i = x_i + b_i$$

and $\tau_1\theta'$ acts trivially on $\mathcal{B}_1 \cup \mathcal{B}_W$. Take, further, an automorphism $\tau_2 \in \Gamma_{(U_2),\{U_1+W\}}$ that interchanges U_1 and W. Then

$$\tau_2^{-1}\tau_1\theta'\tau_2 x_i = x_i + a_i' \quad \forall i \in I$$

and $a_i' \in W$. As above some element τ_3 of $\Gamma_{(U_1),\{U_2+W\}}$ kills all a_i':

$$\tau_3\tau_2^{-1}\tau_1\theta'\tau_2 = \mathrm{id}\,.$$

Thus

$$\theta = \tau_3^{-1}\tau_2^{-1}\tau_1^{-1}\pi\tau_2\pi^{-1}$$

and the length of θ relative to X is at most $3 + 3 + 1 + 3 = 10$. □

Remark 3.3. (i) The upper bound for the width of Aut(A) relative to X that can be extracted from the proof of Proposition 3.1 is of course not sharp. To improve the said bound the following key lemma [5, the proof of Theorem 2.2, part (b)] in the proof of Swan's result can be applied rather than this result itself.

Lemma. *Let $\mathcal{X}$ be a basis of A. Then any automorphism $\varphi \in$* Aut(A) *can be written as a product $\varphi_1\varphi_2\varphi_3$ where each φ_i fixes pointwise a moiety of $\mathcal{X}$ and, moreover, for some moiety $\mathcal{Y}_1$ of $\mathcal{X}$ fixed by φ_1 pointwise the subgroup $\langle \mathcal{X} \setminus \mathcal{Y}_1 \rangle$ is φ_1-invariant.*

(ii) It might be verified quite easily that in fact Swan's proof works in greater generality – in the context of the automorphism groups of free modules. Indeed, no changes in the proof is required to achieve the same conclusion as in Theorem 3.2 for the automorphism groups of infinitely generated free modules over a ring R provided that R satisfies the following condition:

$$\text{for every free module } M \text{ over } R \text{ any direct complement of a free direct summand of } M \text{ of rank one is a free } R\text{-module.} \tag{3.1}$$

Consequently, a stronger result than Proposition 3.1 is true.

Theorem 3.4. *Let R be a ring with the property* (3.1). *Then the variety of all R-modules is a BMN-variety. The automorphism group of any infinitely generated free R-module is a group of universally finite width and satisfies the properties* (i-iii) *listed in Theorem* 2.5.

We turn now to the study of the situation with the automorphism groups of free nilpotent groups.

Theorem 3.5. (i) *Any variety $\mathfrak{N}_c$ of all nilpotent groups of class at most c is a BMN-variety;*

(ii) *if N is an infinitely generated free nilpotent group, then the automorphism group* $\mathrm{Aut}(N)$ *of N is a group of universally finite width and satisfies the properties* (i-iii) *listed in Theorem* 2.5.

Proof. We shall prove the theorem by induction on c. Proposition 3.1 corresponds then to the case when $c = 1$.

Let N be an infinitely generated free nilpotent group of nilpotency class c and let $N_c = \gamma_c(N)$ denote the cth term of the lower central series of N. It is well-known that N_c is a free abelian group and N/N_c is a free nilpotent group of class $c-1$ [17]. If $\mathcal{B}$ is a basis of N then the subgroup N_c is generated by basic commutators

$$[b_1, \ldots, b_c] = [b_1, [b_2, \ldots [b_{c-1}, b_c]] \ldots] \tag{3.2}$$

where $b_1, \ldots, b_c$ are elements of $\mathcal{X}$ [17, Section 5.3].

Assume that

$$N = U_1 * U_2 * W,$$

where U_1, U_2, W are moietous factors of N. Write X for the set

$$\Gamma_{(U_1),\{U_2*W\}} \cup \Gamma_{(U_2),\{U_1*W\}}.$$

According to [3], the homomorphism $\mathrm{Aut}(F) \to \mathrm{Aut}(F/\gamma_k(F))$, where F is a free group induced by the canonical homomorphism $F \to F/\gamma_k(F)$ is surjective for all $k \geqslant 2$. Then the homomorphism $\widehat{\ }: \mathrm{Aut}(N) \to \mathrm{Aut}(N/N_c)$ induced by the canonical homomorphism $N \to N/N_c$ is surjective, too. It is easy to see that the kernel K of the homomorphism $\widehat{\ }$ is abelian.

The image $\widehat{X}$ of X generates $\mathrm{Aut}(N/N_c)$; since by the induction hypothesis the latter group is a group of universally finite width, then

$$\mathrm{Aut}(N/N_c) = \widehat{X}^m$$

for some natural number m. Therefore

$$\mathrm{Aut}(N) = X^m K$$

and it remains to prove that K has finite width relative to X.

As K is abelian, $K = K_{(U_1)} K_{(U_2*W)}$. We show that the lengths of all elements in $K_{(U_1)}$ relative to X are uniformly bounded from above. A similar argument can be then applied to the elements of the subgroup $K_{(U_2*W)}$, which will prove the result.

Suppose that $\mathcal{X}$ is a basis of U_1 and $\mathcal{Y} = \mathcal{Z} \cup \mathcal{T}$, where $\mathcal{Z}$ and $\mathcal{T}$ are bases of U_2 and W, respectively. Let, further,

$$\mathcal{Y} = \mathcal{Y}_1 \cup \ldots \cup \mathcal{Y}_c \cup \mathcal{Y}_{c+1}$$

be a partition of $\mathcal{Y}$ into $(c+1)$ moieties.

Take an arbitrary $\alpha \in K_{(U_1)}$ and let $\mathcal{X} = (x_i : i \in I)$. We have

$$\alpha x_i = x_i t_i \quad \forall i \in I$$

where t_i is in N_c. We write t_i as a product of the basic commutators of the form (3.2) over the basis $\mathcal{X} \cup \mathcal{Y}$ and then rearrange them to obtain a representation

$$t_i = t_{i1} \ldots t_{i,c+1}$$

where t_{ij} is a product of basic commutators of the form (3.2) over the set

$$\mathcal{X} \cup (\mathcal{Y} \setminus \mathcal{Y}_j)$$

(since each basic commutator has only at most c occurrences of elements of $\mathcal{Y}$, there must be a set $\mathcal{Y}_k$ in which none of these elements is contained; this implies that each basis commutator lies in the subgroup $\langle \mathcal{X} \cup (\mathcal{Y} \setminus \mathcal{Y}_m)\rangle$ for an appropriate m.)

For every $k = 1, \dots, c+1$ define the automorphism α_k of N as follows:

$$\begin{aligned} \alpha_k x_i &= x_i t_{ik} \quad \forall i \in I, \\ \alpha_k y &= y \qquad \forall y \in \mathcal{Y} \end{aligned}$$

(note that if (b_k) is a basis of a free nilpotent group G, then any system (b'_k) of elements of G such that b'_k is congruent b_k modulo $[G, G]$ for all k is also a basis of G [18, Theorem 31.25]). We have therefore that $\alpha = \alpha_1 \dots \alpha_{c+1}$.

Clearly, for every $k = 1, \dots, c+1$ the automorphism α_k fixes $\mathcal{Y}_k$ elementwise and preserves the subgroup $\langle \mathcal{X} \cup (\mathcal{Y} \setminus \mathcal{Y}_k)\rangle$. This means that a conjugate of α_k by some automorphism π acting on $\mathcal{X} \cup \mathcal{Y}$ as a permutation is an element of $\Gamma_{(U_1),\{U_2 * W\}}$. Once again by Lemma 2.3 the length of π with respect to X does not exceed 3. Hence the length of α_k relative to X is at most $3+1+3 = 7$. The length of α is therefore at most $7(c+1)$. $\square$

4. Relatively free groups of countably infinite rank

In this section we answer affirmatively Bergman's question about the automorphism groups of free groups of infinite rank for the case of countably infinite rank. The very special feature of the automorphism groups of free groups of *countably* infinite rank is that they are so-called Polish groups, that is, topological groups whose topology is Polish. Recall that a topological space X is *Polish* if it is separable and there is a compatible metric d with respect to which X is a complete metric space (see [13]; a nice introductory account to Polish group can be also found in [12].) Developing ideas from a paper [11] by Hodges–Hodkinson–Lascar–Shelah, Bryant and Evans [2] introduced the notion of automorphisms of relatively free groups that were generic with respect to finitely generated free factors of these groups and proved that under some natural conditions there were 'many' such generic automorphisms.

Consider a relatively free group G of countably infinite rank. As it has been said above $\Gamma = \mathrm{Aut}(G)$ can be viewed as a Polish group: to achieve that one defines a basis of open neighboordhoods of the identity to be the family of all subgroups $\Gamma_{(U)}$, where $\gamma \in \Gamma$ and U is a finite subset of G. A compatible metric d is constructed as follows: if $G = \{a_n : n \in \mathbf{N}\}$ is an enumeration of G, then $d(g, h) = 0$, if $g = h$ and $1/2^n$ otherwise, when n is the minimal natural number such that $ga_n \neq ha_n$ or $g^{-1}a_n \neq h^{-1}a_n$.

Now let us reproduce some definitions and facts from [2]. Write $\mathcal{B}(G)$ for the family of all finitely generated free factors of G. We say that a tuple $(\gamma_1, \dots, \gamma_n)$ of elements of Cartesian power Γ^n of Γ is $\mathcal{B}(G)$-*generic,* if the following two conditions are satisfied.

(1) for all $A \in \mathcal{B}(G)$ the subgroups $\Gamma_{(B)}$ for which $A \subseteq B \in \mathcal{B}(G)$ and $\gamma_i B = B$, for $i = 1, \dots, n$ form a basis of open neighbourhoods of the identity;

(2) if $A \in \mathcal{B}(G)$, $\gamma_i A = A$ for $i = 1, \dots, n$, $A \subseteq B \in \mathcal{B}(G)$, $\beta_1, \dots, \beta_n \in \mathrm{Aut}(B)$ and $\beta \restriction A = \gamma_i \restriction A$ for $i = 1, \dots, n$, then there exists $\alpha \in \Gamma_{(A)}$ such that $\gamma_i^\alpha \restriction B = \beta_i$ for $i = 1, \dots, n$. (Here and in what follows $\gamma^\alpha = \alpha\gamma\alpha^{-1}$.)

Let us equip Cartesian powers Γ^n of Γ with product topologies. Recall that a subset A of a topological space X is called *comeagre* if A contains a countable intersection of open dense sets. Complements of comeagre sets are called *meagre*; it is easy to see that a countable union of meagre sets is again meagre. The group $\Gamma = \mathrm{Aut}(G)$ is said to have *ample* $\mathcal{B}(G)$-*generic automorphisms* if for every natural n the set of elements of Γ^n which are $\mathcal{B}(G)$-generic is comeagre in Γ^n.

Bryant and Evans [2] suggested the following property which implies ampleness of $\mathcal{B}(G)$-generic automorphisms. Let $(y_n : n \in \mathbf{N})$ be a basis of G. Then G has the *basis confinality property* if, for every $\alpha \in \Gamma$ and every $n \in \mathbf{N}$, there exist $r \in \mathbf{N}$ with $r \geqslant n$ and $\beta \in \mathrm{Aut}(\langle y_1, \dots, y_r \rangle)$ such that $\beta y_i = \alpha y_i$ for $i = 1, \dots, n$. Lemma 1.2 from [2] then says that if Γ has basis confinality property, then Γ has ample $\mathcal{B}(G)$-generic automorphisms.

It is easy to see that if G is (absolutely) free, then G has basis confinality property and, hence, ample $\mathcal{B}(G)$-generic automorphisms [2, Lemma 1.4]. Furthermore, the basis confinality property of free groups is inherited by so-callled *tame* automorphism groups of relatively free groups [2, Lemma 1.7]. Let $\mathfrak{V}$ be a variety of group and F a free group of countably infinite rank. Consider the relatively free group $F/\mathfrak{V}(F)$ of $\mathfrak{V}$, where $\mathfrak{V}(F)$ is the verbal subgroup of F corresponding to $\mathfrak{V}$. Then the automorphism group $\mathrm{Aut}(F/\mathfrak{V}(F))$ is called *tame*, if the natural homomorphism $\mathrm{Aut}(F) \to \mathrm{Aut}(F/\mathfrak{V}(F))$ induced by the canonical homomorphism $F \to F/\mathfrak{V}(F)$ is surjective. For instance, for the varieties $\mathfrak{V}$ such that the relatively free group $F/\mathfrak{V}(F)$ is nilpotent, the automorphism group $\mathrm{Aut}(F/\mathfrak{V}(F))$ is tame ([3]; see [2, Theorem 1.6] for the other examples).

We shall use the following notation: if α is an ordinal, then ${}^{\alpha}2$ is the set of all functions from α onto $2 = \{0, 1\}$. The symbol ${}^{<\omega}2$ denotes the set $\bigcup_{n<\omega} {}^{n}2$. Notation $\sigma < \tau$, where σ, τ are functions means that τ is an extension of σ. If $s \in {}^{n}2$, then as in [11] $s\frown 0$ (resp. $s\frown 1$) is the extension of s to the ordinal $n+1$ which takes the value 0 (resp. 1) at n.

Theorem 4.1. *Let G be a relatively free group of countably infinite rank which has ample $\mathcal{B}(G)$-generic automorphisms. Then* $\mathrm{Aut}(G)$ *is a group of universally finite width.*

Proof. Suppose, towards a contradiction, that $\Gamma = \mathrm{Aut}(G)$ has infinite width relative to a generating set X. By including the identity automorphism of G into X, if necessary, we may assume that the system (X^m) of powers of X forms a chain. Repeating the argument from the sufficiency part in the proof of Theorem 2.4 one sees that some power X^{k_0} of X contains a pair of stabilizers

$$\Gamma_{(U_1),\{U_2 * W\}} \text{ and } \Gamma_{(U_2),\{U_1 * W\}},$$

where U_1, U_2, W are moietous free factors with $G = U_1 * U_2 * W$. Then by Lemma 2.3 X^{3k_0} contains the subgroup of all automorphisms of G that act as permutations on a certain basis $\mathcal{X}^*$ of G.

Our next task is to make necessary preparations for a 0-1 game with $\mathcal{B}(G)$-generic automorphisms of G (here we use the word 'game' in the literal rather than in the formal sense, though in fact some game-theoretic argument may be carried out, see Remark 4.2.4 in [11]) The reader is referred to the proof of Theorem 5.3 in [11] where the game has been first described for the so-called ample homogeneous generic automorphisms in the automorphism groups of ω-stable ω-categorical structures and for the automorphism group of the random graph. Later Bryant and Evans [2] demonstrated that the game can be adapted for the $\mathcal{B}(G)$-generic automorphisms of G, thereby proving that $\mathrm{Aut}(G)$ had the small index property and uncountable confinality.

Let us briefly discuss one of the versions of the original game Bryant and Evans applied in their paper. Let $\{a_n : n < \omega\}$ be enumeration of G and (H_n) a countable chain of proper subgroups of $\mathrm{Aut}(G)$ whose union is $\mathrm{Aut}(G)$. Using induction on $s \in {}^{<\omega}2$ one constructs of a free factor $B_s \in \mathcal{B}(G)$ and elements $\gamma_s, g_{s\frown 0}, g_{s\frown 1} \in \mathrm{Aut}(G)$ which satisfy the following five conditions (see [11, p. 216]):

G1. $\gamma_\varnothing = \mathrm{id}$ and if $t \in {}^{<\omega}2$, $t \leqslant s$ and $t \neq \varnothing$, then $g_t B_s = B_s$;
G2. if $s \in {}^{n}2$, then $g_{s\frown 0} \in \Gamma_{(B_s)} \cap H_n$ and $g_{s\frown 1} \in \Gamma_{(B_s)} \setminus H_n$;
G3. if $s \in {}^{n}2$ with $n > 0$, then the tuple $\overline{g}_s = (g_{s\restriction 1}, \dots, g_{s\restriction n})$ is a $\mathcal{B}(G)$-generic;
G4. $(g_t)^{\gamma_s} = (g_t)^{\gamma_t}$ for all t such that $\varnothing < t \leqslant s$;
G5. if $s \in {}^{n}2$ then $\gamma_{s\frown 0}\gamma_s^{-1}, \gamma_{s\frown 1}\gamma_s^{-1} \in \Gamma_{(a_i)} \cap \Gamma_{(\gamma_s^{-1} a_i)}$ for all $i \leqslant n$.

The induction argument relies heavily on ampleness of $\mathcal{B}(G)$-generic automorphisms, since $g_{s\frown 0}$ and $g_{s\frown 1}$ are taken actually from sets $(C \cap \Gamma_{(B_s)}) \cap H_n$ and $(C \cap \Gamma_{(B_s)}) \setminus H_n$ respectively, where

C is a suitable comeagre set of $\mathcal{B}(G)$-generic automorphisms. Changing the second condition G2, the 0-1 condition of the game, one can achieve different goals: for instance, to prove (by contradiction) that a subgroup H of Γ of small index is open one changes the second condition to

G2′. $g_{s\frown 0} \in \Gamma_{(B_s)} \cap H$ and $g_{s\frown 1} \in \Gamma_{(B_s)} \setminus H$

Our version of the game assumes the following variant of the condition G2:

G2″. if $s \in {}^n2$, then $g_{s\frown 0} \in \Gamma_{(B_s)} \cap X^n$ and $g_{s\frown 1} \in \Gamma_{(B_s)} \setminus X^{3n}$.

Lemma 4.2. *There is a natural m_0 such that for every $m \geqslant m_0$ and for every finite set E of G*

(i) $\Gamma_{(E)} \cap X^m$ *is not comeagre in* $\Gamma_{(E)}$;

(ii) $\Gamma_{(E)} \cap X^m$ *is not meagre;*

(iii) *for every comeagre set C of Γ both sets*

$$(C \cap \Gamma_{(E)}) \cap X^m \text{ and } (C \cap \Gamma_{(E)}) \setminus X^m$$

are not empty.

Proof. Note that $\Gamma_{(E)}$ is a Polish group. We shall need one result from a paper [4] which is an immediate corollary of Lemma 2.1 from [4].

Lemma 4.3. *Let $\mathcal{X}$ be any basis of G and E a finite subset of $\mathcal{X}$. Then any automorphism $\alpha \in \Gamma$ can be written as a product $\beta\gamma$, where $\beta \in \Gamma_{(E)}$ and $\gamma \in \Gamma_{(V_1),\{V_2\}}$, where V_1 and V_2 are subgroups generated by disjoint moieties of $\mathcal{X}$ whose union is $\mathcal{X}$.*

(i) Suppose otherwise: let $Y_m = \Gamma_{(E)} \cap X^m$ be comeagre in $\Gamma_{(E)}$ for some finite $E \subset G$. Take $\gamma \in Y_m$. Since Y_m is comeagre in $\Gamma_{(E)}$, the set γY_m is also comeagre. Any two comeagre subsets always have a non-empty interesection, and then $\gamma Y_m \cap Y_m \neq \varnothing$ which means that

$$\gamma \in Y_m Y_m^{-1} \subseteq X^{2m},$$

whence $\Gamma_{(E)} \subseteq X^{2m}$. Without loss of generality we may assume that E is a subset of the above-defined basis $\mathcal{X}^*$. Any subgroup of the form $\Gamma_{(V_1),\{V_2\}}$ from Lemma 4.3 over $\mathcal{X}^*$ is evidently conjugate to the subgroup $\Gamma_{(U),\{W\}}$ by a suitable element of Γ that acts on $\mathcal{X}^*$ as a permutation. Thus by Lemma 4.3 we have

$$\Gamma \subseteq X^{2m} X^{3k_0} X^{k_0} X^{3k_0} = X^{2m+7k_0},$$

a contradiction.

(ii) As Γ is Polish and $\Gamma = \bigcup X^m$, all but finitely many of the elements of the chain (X^m) are meagre. Take as n_0 the minimal $m \geqslant 7k_0$ such that X^m is not meagre. Similarly to (i), Lemma 4.3 implies that

$$\Gamma = \Gamma_{(E)} X^{7k_0} = \Gamma_{(E)} X^{n_0}. \tag{4.1}$$

The index of the open subgroup $\Gamma_{(E)}$ in Γ is at most ω. By (4.1) there is a complete system $\{z_j : j \in J\}$ where $|J| \leqslant \omega$ of representatives of left cosets $\gamma\Gamma_{(E)}$ which consists only of elements of X^{n_0}. Let

$$Z_j = \{z \in X^{n_0} : z \equiv z_j (\mod \Gamma_{(E)})\}$$

where $j \in J$. Since $X^{n_0} = \bigcup_{j \in J} Z_j$, then one of the sets Z_j, say Z_{j_0} is not meagre. The translate $z_{j_0}^{-1} Z_{j_0}$, a subset of X^{2n_0}, is also not meagre. We then have

$$z_{j_0}^{-1} Z_{j_0} \subseteq \Gamma_{(E)} \cap X^{2n_0}$$

and hence $\Gamma_{(E)} \cap X^m$ is not meagre provided that $m \geqslant 2n_0$.

(iii) Every comeagre set meets every nonmeagre set, and then

$$(C \cap \Gamma_{(E)}) \cap X^m = C \cap (\Gamma_{(E)} \cap X^m) \neq \varnothing$$

by (ii).

The set $C^* = C \cap \Gamma_{(E)}$ is comeagre in $\Gamma_{(E)}$. Then by (i) the relation $C^* \subseteq X^m$ is impossible. $\square$

Without loss of generality we may assume that every power of X satisfies the condition (iii) of Lemma 4.2. Following the original game in [11, p. 216–217] the reader make check now that the properties of the $\mathcal{B}(G)$-generic automorphisms listed above and Lemma 4.2 (iii) makes it possible to construct elements $g_{s\frown 0}, g_{s\frown 1}$ and γ_s, where $s \in {}^n2$ satisfying the conditions G1,G2$''$,G3,G4,G5.

We continue by analogy with the proof of Theorem 6.1 in [11] making necessary adjustments. Suppose $\sigma \in {}^\omega 2$. The condition G5 implies that the sequence $(\gamma_{\sigma\restriction n})$ is Cauchy; let γ_σ denote its limit. It then follows from G4 that

$$(g_{s\frown 0})^{\gamma_\sigma \gamma_\tau^{-1}} = g_{s\frown 1} \quad \forall \sigma > s\frown 0, \forall \tau > s\frown 1 \tag{4.2}$$

where σ, τ are in ${}^\omega 2$. [11, pp. 216–216]. Suppose now that $s \in {}^n2$. By the construction $g_{s\frown 0}$ is X^n, while $g_{s\frown 1}$ is not in X^{3n}. The equation (4.2) implies then that $\gamma_\sigma \gamma_\tau^{-1}$ does not belong to X^n, because the length of this element with regard to X must be greater than n. In particular, if $\sigma \neq \tau$, then $\gamma_\sigma \neq \gamma_\tau$, since the length of $\gamma_\sigma \gamma_\tau^{-1}$ relative to X is greater than 1.

Thus there are 2^ω elements of the form γ_σ and hence some power X^n of X contains uncountably many of them:

$$X^n \supseteq \{\gamma_\sigma : \sigma \in \Sigma\}$$

where Σ is an uncountable subset of ${}^\omega 2$. There exists however $m \geqslant 2n$ such that for some $\sigma, \tau \in \Sigma$ we have

$$\sigma \restriction m = \tau \restriction m \text{ and } \sigma \restriction m+1 \neq \tau \restriction m+1.$$

But as we saw above it follows that $\gamma_\sigma \gamma_\tau^{-1}$, a product of elements of X^m and also an element of X^{2n}, does not belong to X^m, a contradiction.

$\square$

Corollary 4.4. *Let F be a free group of countably infinite rank. Then* $\mathrm{Aut}(F)$ *is a group of universally finite width.*

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